# NOTES ON A POSTHUMOUS PAPER BY F. HAHN

BY

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ABSTRACT

This is a short exposition and slight reorgnization of Hahn's work concerning discrete real time flows with quasi-discrete spectra.

The paper [1], I refer to (Discrete real time flows with quasi-discrete spectra and algebras generated by exp $q(t)$) consists of notes (essentially unedited) sent to me in 1964. There seemed to be insurmountable gaps in Hahn's work at that time, which led him to abandon his project. However, the first two sections of his paper were complete and a letter of that time indicated his intention to publish them. Evidently he did not get around to it, which is surprising as the theory is interesting and is a continuation of his work in [2].

This note depends on [1], as its purpose is primarily to suggest a suitable organisation of Hahn's theory as set out in §§3–6. Some minor simplifications are provided and since his ideas are applicable to actions of subgroups of the reals, we present the theory of quasi-discrete spectrum for such groups.

Concerning the last remarks in §7 of [1], Hahn was mistaken in believing total minimality played no role. Throughout there are assumptions that $H_1$ is torsion free, for some such assumption is required to prove that quasi-eigenfunctions are orthogonal. Explicitly we shall assume that flows are totally minimal.

## 1. Preliminaries

Let $X$ be a compact Hausdorff space and let $T_t$ be a homeomorphism for each $t \in R \subset \mathbb{R}$ ($R$ is a subgroup of the reals) where $T_{t+s} = T_t \circ T_s$. In other words we have a discrete $R$-flow $T_t$ on $X$. If $K = \{z: |z| = 1\}$ is the circle group, $C(X, K)$

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is the multiplicative groups of continuous functions with values in $K$. We shall *always* assume that when $fT_t = f$ for all $t$ in a syndetic subgroups of $R$ ($f \in C(X,K)$) then $f$ is constant, which is the case for example when $T$ is totally minimal. ($R' \subset R$ is *syndetic* means that $R/R'$ is finite).

If $f \in C(X,K)$ is a $T$ eigenfunction then $fT_t = \lambda(t)f$ for all $t \in R$ where $\lambda: R \to K$ is a homomorphism, i.e., $\lambda \in \hat{R}$ (the character group of $R$).

Let $G_1$ be the group of eigenfunctions in $C(X,K)$ and let $H_1 = \{fT_t/f : f \in G_1\}$ be the group of eigenvalues. If $\hat{R} \supset H_1 \subset H_2 \subset \cdots \subset H_{n-1}$

$$G_1 \subset G_2 \subset \cdots \subset G_{n-1}$$

have been defined where $G_i \subset C(X,K)$ and $H_i$ consists of continuous maps: $R \times X \to K$ and $g \in H_i$ implies $g_t \in G_{i-1}$ ($g_t(x) = g(t,x)$) then $G_n$, $H_n$ are defined as follows:

$$G_n = \{f \in C(X,K): fT_t/f \in G_{n-1}\}$$
$$H_n = \{g : g(t,x) = f(T_t x)/f(x),\ f \in G_n\}.$$

Let $G = \bigcup_n G_n$, $H = \bigcup_n H_n$ be the groups of *quasi-eigenfunctions* and *quasi-eigenvalues* respectively.

The system $(X,T)$ is said to have *quasi-discrete spectrum* if $G$ spans $C(X)$ or, what amounts to the same thing, if $G$ separates points of $X$.

Let $\rho$ denote the set of homomorphisms of $H$ into $H$ given by (for each $s \in R$),

$$(\rho_s g)(t,x) = \frac{g(t,T_s x)}{g(t,x)}.$$

As Hahn shows in §3 of [1]:

$$\rho_s \circ \rho_t g = (\rho_s g)^{-1}(\rho_t g)^{-1}\rho_{s+t} g = \rho_t \circ \rho_s g. \tag{1.1}$$

Hahn defined an abstract system of quasi-eigenvalues $(\{A_n\},\sigma)$ as follows: $\hat{R} \supset A_1 \subset A_2 \subset \cdots$ are abelian groups with $A_1$ torsion free, $\sigma: R \to \hom(A)$, $A = \bigcup_n A_n$, $\sigma_t: A_n \to A_{n-1}$ satisfies (1.1) and $\ker \sigma_t = A_1$ (if $t \neq 0$, $t \in R$).

He made the following assumption (which amounts to a secondary gap, for it is not clear whether it is always satisfied):

(1.2) If $\alpha(t)$ is the character of $A_1$ ($t \in R$) defined by $\langle \alpha(t), \lambda \rangle = \lambda(t)$, then there is a homomorphism extension $\gamma(t)$ of $\alpha(t)$ to all of $A$, for each $t \in R$, such that $\langle \gamma(t+s), a \rangle = \langle \gamma(t)\gamma(s), a \rangle \langle \gamma(s), \sigma_t a \rangle$ for all $t$, $s \in R$.

The major gap in Hahn's work is related to the following assumption:

(1.3) If $\gamma(t)$, $\gamma'(t)$ are two such extensions of $\alpha(t)$ then there exists a homomorphism $C: A \to K$ such that

$$\langle c, \sigma_t a \rangle = \left( \frac{\gamma(t)}{\gamma'(t)}, a \right) \text{ for all } t \in R.$$

In other words we have existence and "uniqueness" assumptions. (1.2), (1.3) will be explicitly mentioned when they are needed.

EXISTENCE THEOREM. *If* $(\{A_n\}, \sigma)$ *is a system of quasi-eigenvalues (satisfying* (1.2)) *then there is a totally minimal* $R$ *flow* $T$ *on a compact Hausdorff space* $X$ *with quasi-discrete spectrum whose system of quasi-eigenvalues* $(\{H_n\}, \rho)$ *is equivalent to* $(\{A_n\}, \sigma)$, *i.e., there is an isomorphism* $\phi$ *between* $H$ *and* $A$ *which restricts to an isomorphism between* $H_n$ *and* $A_n$ *(and restricts to the identity on* $H_1 \equiv A_1$) *such that*

$$\begin{array}{ccc} H_n & \xleftarrow{\rho_t} & H_{n+1} \\ \phi \downarrow & & \downarrow \phi \\ A_n & \xleftarrow[\sigma_t]{} & A_{n+1} \end{array}$$

*commutes for all* $t \in R$.

REPRESENTATION THEOREM. *If* $(X, T)$ *is a totally minimal discrete* $R$ *flow with quasi-discrete spectrum then there is a compact connected abelian* $X'$ *and a unipotent (see later) flow of affines* $T'_t (t \in R)$ *such that* $(X, T)$, $(X', T')$ *are topologically conjugate.*

UNIQUENESS THEOREM. *If* $(X, T)$, $(X', T')$ *are totally minimal discrete* $R$ *flows with quasi-discrete spectra and with equivalent systems of quasi-eigenvalues, then for each* $t \in R$ $(X, T_t)$, $(X', T_t)$ *are topologically conjugate. (If* $(H, \rho)$ *or* $(H', \rho')$ *satisfies* (1.3) *then there is a "global" conjugacy between* $(X, T)$ *and* $(X', T')$.)

In any case one can prove:

RIGIDITY THEOREM. If $(X, T), (X', T')$ *are totally minimal* $R$ *flows of unipotent affnes with quasi-discrete spectra on compact abelian groups and if* $\phi$ *is a topological conjugacy between them then* $\phi$ *is affine.*

## 2. Existence theorem

Hahn's treatment of this theorem is perfectly adequate, but perhaps a few words are in order. The system $A_1 \overset{i}{\subset} A_2 \overset{i}{\subset} A_3 \cdots$, $i$ inclusion, gives rise to a dual system, $X_1 \overset{i}{\leftarrow} X_2 \overset{i}{\leftarrow} X_3 \cdots$ with a projective limit $X$ which is compact Hausdorff. Moreover the homomorphisms

$$A_1 \underset{\sigma_t}{\leftarrow} A_2 \underset{\sigma_t}{\leftarrow} A_3$$

give rise to homomorphisms

$$X_1 \underset{\sigma_t}{\rightarrow} X_2 \underset{\sigma_t}{\rightarrow} X_3$$

which are induced by homomorphisms $\hat{\sigma}_t: X \to X$. If $S_t x = x \cdot \hat{\sigma}_t x$, then $S_t$ $(t \in R)$ is a group of automorphisms of $X$. The group $S(S_t, t \in R)$ is *unipotent* in the sense that $\{\hat{\sigma}_t: t \in R\}$ is *nilpotent*, i.e., for each $\eta \in A \simeq \hat{X}$, $\eta(\hat{\sigma}_{t_1} \circ \cdots \circ \hat{\sigma}_{t_n}) \equiv 1$ for all $t_1, \cdots t_n \in R$. ($n$ may depend on $\eta$.) (2.2) is invoked to provide translations $\gamma(t) \in X$ for each $t \in R$ so that $T_t(x) = \gamma(t) S_t x$, becomes an $R$ flow of unipotent affines. Unipotence *guarantees* that $T(T_t, t \in R)$ has quasi-discrete spectrum since each character of $X$, is a quasi-eigenfunction. A is torsion free for if $a^n = 1$, $a \in \ker \sigma_t^k - \ker \sigma_t^{k-1}$ then $\sigma_t^{k-1} a \in A_1$ and $\sigma_t^{k-1}(a^n) = 1$ so that $\sigma_t^{k-1}(a) = 1$ (since $A_1$ is torsion free). Hence $a \in \ker \sigma_t^{k-1}$ contradicting the definition of $k$. Consequently $X$ is connected.

We shall show that $\{T_t: t \in R\}$ is ergodic with respect to Haar measure on $X$.

If $f \in L^2(X)$, $U_t f = f$ for each $t \in R$ where $U_t g = g \circ T_t$, then $f$ has a Fourier series, $\Sigma_\eta a(\eta)\eta = \Sigma_\eta a(\eta) U_t \eta$ for each $t \in R$. $\Sigma_\eta a(\eta)\eta = \Sigma_\eta a(\eta)\eta(\gamma(t))\eta \circ S_t$ so that

$$a(\eta) \cdot \eta(\gamma(t)) = a(\eta(S_t)), \text{ i.e., } |a(\eta)| = |a(\eta \circ S_t)|$$

for all $t \in R$. Of course $a(\eta) = 0$ for all but countably many $\eta$, and if $a(\eta) \neq 0$ then $\eta(S_t) = \eta$ for a syndetic subgroup $R'$ of $R$ otherwise we should have infinitely many non-zero coefficients with the same modulus. But this implies $\eta(\gamma(t)) = 1$ for all $t \in R'$. Hence $\eta T_t = \eta$ for all $t \in R'$. Since $R'$ is syndetic and $X$ is connected, it follows that $\eta T_t = \eta$ for all $t \in R$. Hence $\eta$ corresponds to a member of $A_1$ and $\langle \eta, \gamma(t) \rangle = \langle \eta, \alpha(t) \rangle = \eta(t) \equiv 1$, i.e., $\eta$ is trivial. This shows that $f$ is constant and $T$ is ergodic.

Since $T$ is distal (see [1]) $T$ is minimal and since $X$ is connected it follows that $T$ is totally minimal. (The argument above could also be adapted to show that $T$ is totally ergodic.)

The proof that $(\{H_n\},\rho)$ is equivalent to $(\{A_n\}, \sigma)$ is straightforward as we know $H_n$ explicitly:

$$H_n = \{\eta(\gamma(t))\eta(x^{-1}S_t x) : \eta \in \hat{X},$$

$$\eta(\hat{\sigma}_{t_1} \circ \cdots \circ \hat{\sigma}_{t_n}) \equiv 1 \text{ for all non-zero } t_1, \cdots t_n \in R\}.$$

Indeed these are the $n$th order quasi-eigenvalues corresponding to the $n$th order quasi-eigenfunctions:

$$G_n = \{\eta \in \hat{X} : \eta(\hat{\sigma}_{t_1} \circ \cdots \circ \hat{\sigma}_{t_n}) \equiv 1 \text{ for all non-zero } t_1, \cdots t_n \in R\}.$$

That these are *all* of the $n$th order quasi-eigenfunctions, follows from the fact that quasi-eigenfunctions are orthogonal (see next section) and characters separate points.

## 3. Representation theorem

Let $(X, T_t)$ be a totally minimal $R$ flow on a compact Hausdorff space with quasi-discrete spectrum. Let $m$ be a $T_t$ invariant ergodic normalized Baire measure.

THEOREM. *If $f, f' \in G$ then either $f$ is a constant multiple of $f'$ or $\int f\bar{f}'dm = 0$. Hence $T_t$ is uniquely ergodic.*

PROOF. We have to show that for each non-constant $f \in G_n$, $n = 1, 2, \cdots$ $\int f dm = 0$. If $f \in G_1$ ($f$ not constant) then $fT_t = \lambda(t) f$ with $\lambda$ non-trivial. Integration shows that $\int f dm = 0$. Suppose elements of $G_{n-1}$ are either constant multiples of one another or mutually orthogonal, i.e., if $f \in G_{n-1}$ ($f$ non-constant) then $\int f dm = 0$.

Let $f \in G_n$ (non-constant) then

$$fT_t = k(x,t)f \tag{3.1}$$

with $k(x,t) \in G_{n-1}$ for all $t \in R$. Let $\mathscr{A}$ be the smallest $\sigma$-algebra with respect to which elements of $G_{n-1}$ are measurable. Evidently $T_t \mathscr{A} = \mathscr{A}$ and

$$E(f|\mathscr{A}) \circ T_t = k(x,t)E(f|\mathscr{A}). \tag{3.2}$$

Comparing (3.1) and (3.2) we see that ergodicity implies $E(f|\mathscr{A}) = 0$ and $\int f dm = 0$ or $f$ is $\mathscr{A}$ measurable. In the latter case $f$ has an orthogonal expansion $f = \Sigma_n a_n f_n$ with $f_n \in G_{n-1}$ so that $\Sigma_n a_n f_n T_t = fT_t = \Sigma_n a_n k(x,t) f_n$.

Let $f_n T_t = k_n(x,t) f_n$, $k_n(x,t) \in H_{n-1}$, then

$$\sum_n a_n k(x,t) f_n = \sum_n a_n k_n(x,t) f_n$$

and if $b$ is a non-zero coefficient then $N = \{n : a_n = b\}$ is a finite set and

$$\sum_{n \in N} k(x,t) f_n = \sum_{n \in N} k_n(x,t) f_n$$

for all $t \in R$. Hence $f_n k(x,t) = k_{n_t}(x,t) f_{n_t}$ for some $n_t \in N$, i.e., $f_n k(x,t) = k_m(x,t) f_n$ for all $t \in R_m$ where $\{R_m : m \in N\}$ is a partition of $R$. Therefore

$$\frac{f T_t / f_m T_t}{f / f_m} = \frac{f_m}{f_n}$$

for all $t \in R_m$ and it follows that $R_n$ is a subgroup of $R$ with each $R_m$, $m \in N$, a coset of $R_n$. Hence $(f/f_n)T_t = (f/f_n)$ for all $t$ belonging to a syndetic subgroup of $R$ and $f/f_n$ is constant. It follows that $\int f\,dm = 0$.

We have shown that each ergodic measure assigns the same value to a separating group $G$. Thus there is only ergodic measure.

THEOREM. *If $(X,T)$ is totally minimal with quasi-discrete spectrum then $(X,T)$ is topologically conjugate to a flow of affines on a compact abelian group.*

PROOF. Let $G$ be the group of quasi-eigenvalues and let $m$ be *the* $T_t$ invariant normalized Baire measure. $G \supset K$ a divisible group so that $G = \Phi . K$, for some subgroup $\Phi \subset G$, $\Phi \cap K = \{1\}$. Let $X'$ be the compact abelian character group of $\Phi$. If $\phi \in \Phi$ then $\phi \circ T_t \in G$ so that $\phi \circ T_t = \gamma_t(\varnothing) . S_t \phi$ where $\gamma_t(\phi) \in K$, $S_t \phi \in \Phi$. It is immediate that $S_t$ is a one-parameter group of automorphisms of $\Phi$ and $\gamma_t : \Phi \to K$ is a homomorphism for each $t \in R$. Moreover

$$\begin{aligned} \phi \circ T_{t+s} = \gamma_{t+s}(\phi) \cdot S_{t+s}(\phi) &= \gamma_t(\phi) \cdot (S_t \phi) \circ T_s \\ &= \gamma_t(\phi) \gamma_s(S_t \phi) \cdot S_{t+s} \phi \end{aligned}$$

so that $\gamma_{t+s} = \gamma_t \cdot \hat{S}_t \gamma_s$ where $\hat{S}_t$ is the dual of $S_t$. If $T_t'(x') = \gamma_t \cdot \hat{S}_t(x')$ then $T_t'$is a one-parameter group of affines on $X'$. Moreover identifying $\Phi$ with $\hat{X}'$ we see that $T_t$ and $T_t'$ induce the same action on $K \cdot \Phi = K \cdot \hat{X}'$. The identification of $\Phi$ and $\hat{X}'$ extends uniquely to an algebraic (conjugate) isomorphism between the linear span of $G$ and the linear span of $\hat{X}'$. This extension is an isometry between these subspaces of $L^2(X)$ and $L^2(X')$. Using the fact that for each $f \in C(X)$, $\sup_{x \in X} |f(x)| = \lim_{n \to \infty} (\int |f|^{2^n} dm)^{1/2^n}$ we see that this extension is an isometry between these subspaces of $C(X)$ and $C(X')$.

We conclude that there is a (conjugate) algebraic isometry between $C(X)$ and $C(X')$ which sends the action of $T_t$ to the action of $T_t'$. Hence there is a homeomorphism $\psi$ between $X'$ and $X$ with $\psi T_t' = T_t \psi$ for all $t \in R$.

**4. Uniqueness theorem**

THEOREM. *If $(X,T)$, $(X',T')$ are totally minimal R flows with quasi-discrete spectra and with equivalent systems of quasi-eigenvalues then $(X,T_t)$, $(X',T_t')$ are topologically conjugate.*

PROOF. In fact by the representation theorem we may suppose $(X,T_t)$, $(X',T_t')$ are flows of affines on compact abelian groups. Moreover if $G$, $H$, $G'$, $H'$ are the groups of quasi-eigenfunctions and quasi-eigenvalues for $T_t$, $T_t'$ respectively, then $G = K\cdot\hat{X}$ and $G' = K\cdot\hat{X}'$. Let

$$G_n = \{k\cdot\eta\colon k\in K,\ \eta(\sigma_{t_1}\circ\cdots\circ\sigma_{t_n}) \equiv 1,\ \eta\in\hat{X} \text{ for all } t_1,\cdots t_n\in R\}$$

where $T_tx = \gamma(t)\cdot S_tx$ and $\sigma_t(x) = x^{-1}S_tx$. Then $G_n$ consists precisely of the quasi-eigenfunctions of order $n$ and

$$H_n = \{\eta(\gamma(t))\cdot\eta(\sigma_tx)\colon \eta\in G_n\cap\hat{X}\}.$$

Since $(\{H_n\},\rho)$ and $(\{H_n'\},\rho')$ are equivalent by an isomorphism* which restricts to the identity on $H_1$ we see that

$$[\eta(\gamma(t))\eta(\sigma_tx)]^* = \eta'(\gamma'(t))\eta'(\sigma_t'x')$$

for some $\eta'\in\hat{X}'$. Indeed comparing this with another $\eta''\in\hat{X}'$ we conclude that $\eta'/\eta''$ annihilates $\gamma'(t)$ for all $t\in R$ and $\sigma_t'x'$ so that $\eta' = \eta''$. Hence we have a map $\hat{X}\xrightarrow{\prime}\hat{X}'$ which is well defined and easily seen to be an isomorphism.

For each $t$, $s\in R$

$$\begin{aligned}[\eta(\sigma_t\gamma(s))\eta(\sigma_t\sigma_sx)] &= \eta'(\sigma_t'\gamma'(s))\eta'(\sigma_t'\sigma_s'x')\\ &= (\eta\sigma_t)'(\gamma'(s))\cdot(\eta\sigma_t)'(\sigma_s'x'),\end{aligned}$$

since $(\rho_sf)^* = \rho_s'f^*$.

Hence $\eta'\sigma_t' = (\eta\sigma_t)'$ and since $'$ is an isomorphism, $\eta'(\hat{\sigma}_t)\eta' = (\eta\sigma_t\cdot\eta)'$, i.e., $\eta'S_t' = (\eta S_t)'$.

We wish to show that $(\eta\gamma(s)\cdot\eta\sigma_s)^* = C(\eta\sigma_s)\cdot\eta\gamma(s)\cdot(\eta\sigma_s)'$ where $C$ is a homomorphism of $\hat{X}$ to $K$. In fact we require

$$C(\eta\sigma_s)\cdot\eta\gamma(s)\cdot\eta'\sigma_s' = \eta'\gamma'(s)\cdot\eta'\sigma_s'$$

$$\text{i.e., } C(\eta\sigma_s) = \frac{\eta'\gamma'(s)}{\eta\gamma(s)}.$$

If $\eta\sigma_s=\lambda\sigma_s$ then $\eta/\lambda\in G_1$ so that $(\eta/\lambda)(\gamma(t))=(\eta'/\lambda')(\gamma'(t))$ for all $t\in R$. Hence the maps $(\hat{\sigma}_s\hat{X})\xrightarrow{C} K$ defined by

$$\eta\sigma_s \rightarrow \frac{\eta'\gamma'(s)}{\eta\gamma(s)}$$

is an unambiguously defined homomorphism (depending on $s$) which extends to a homomorphism of $\hat{X}$ into $K$ satisfying our requirements.

It is a routine matter then to check that the map $K \cdot \hat{X} \rightarrow K \cdot \hat{X}'$ given by $k \cdot \eta \rightarrow k \cdot C(\eta) \cdot \eta'$ conjugates $T_s$ and $T_s'$ and is induced by an affine. This completes the proof.

As we have just seen the conjugacies in the last theorem depend on each parameter $t \in R$.

A universal or global conjugacy could be constructed if (1.3) is assumed. Even without (1.3), a conjugacy for each subgroup of $R$ of rank 1 can be constructed, by simply using the compactness of $X$. We omit these details and conclude with:

RIGIDITY THEOREM. *If $\phi$ is a topological conjugacy between two totally minimal $R$ flows of unipotent affines then $\phi$ is affine.*

PROOF. Each group of quasi-eigenfunctions consists precisely of constant multiples of characters invariant under the flows. $\phi$ sends quasi-eigenfunctions $K \cdot \hat{X}$ to quasi-eigenfunctions $K \cdot \hat{X}'$. Hence $\eta\phi = C(\eta)\eta'$ where $C(\eta) \in K$, $\eta' \in \hat{X}'$, and $C$ is a homomorphism of $\hat{X}$ to $K$ and $\eta \rightarrow \eta'$ is an isomorphism between $\hat{X}$ and $\hat{X}'$. The proof is completed by showing that such a map is induced by an affine and this is routine.

## REFERENCES

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